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Hamilton cycles in the path graph of a set of points in convex position

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Abstract

Let P be a set of n points in convex position in the plane. The path graph $G(P)$ of P is the graph with one vertex for each plane spanning path of P, in which two paths S and T are adjacent if one can be obtained from the other by a single edge exchange. We prove that if $n \geqslant 3$, then $G(P)$ is hamiltonian. © 2001 Published by Elsevier Science B.V.

Keywords: Geometric graph; Spanning path

1. Introduction

For any connected abstract graph G, the *tree graph* $T(G)$ is the graph that has one vertex for each spanning tree of G, and an edge joining trees R and S whenever R can be obtained from S by deleting an edge s of S and adding an edge r of R. Cummings proved in [2] that $T(G)$ is hamiltonian; see also [4] for a short proof.

A geometric variation that has been studied is the following: For a set P of points in general position in the plane, the *plane tree graph* $T(P)$ of P is defined as the abstract graph with one vertex for each plane spanning tree of P, in which two trees are adjacent if, as in the abstract case, one is obtained from the other by a single edge exchange. Avis and Fukuda proved in [1] that $T(P)$ is always connected. In [3], Hernando et al. show that if the points in P are the vertices of a convex polygon, then $T(P)$ is hamiltonian.

In this note we only consider sets P of points in convex position and study the subgraph $G(P)$ of $T(P)$ induced by the set of plane spanning paths of P. We prove that $G(P)$ is itself hamiltonian.

Since, for any spanning path T of P, planarity depends only on the relative position of its vertices along the convex hull of P, for any set P of n points in convex position in the plane the graph $G(P)$ is

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Fig. 1. The graphs G_3 and G_4.

isomorphic to $G(P_n)$, where P_n is a regular n-gon. We denote by G_n the graph $G(P_n)$; the graphs G_3 and G_4 are shown in Fig. 1.

The main result of this article is the following.

Theorem 1. *If $n \geqslant 3$, then G_n is hamiltonian.*

For any geometric graphs G and F on the same set of points we say that G and F are *adjacent* if F can be obtained from G by a single edge exchange.

Throughout the paper $w_1, w_2, \ldots, w_n$ denote the vertices of P_n in clockwise order. For any vertices w_t and w_s of P_n we denote by $[w_t, w_s]$ the set $\{w_t, w_{t+1}, \ldots, w_s\}$, where indices are added modulo n.

2. Preliminary results

A *leaf* of a tree T is an edge incident with a vertex of T with degree one. For any set P of points in the plane we denote by $CH(P)$ the boundary of the convex hull of P. In [3], Hernando et al. proved the following result.

Theorem 2. *Let P be a set of $n \geqslant 3$ points in convex position in the plane. If T is a plane spanning tree of P, then T has at least two leaves lying in $CH(P)$.*

Corollary 3. *Let P be a set of n points in convex position in the plane. If e is a leaf of a plane spanning path T of P, then e lies in $CH(P)$.*

Proof. If $n \leqslant 3$, then all edges of T lie in $CH(P)$. And if $n \geqslant 3$, then T has exactly two leaves which, by Theorem 2, must lie in $CH(P)$. $\square$

Let $T = w_1, w_{t_2}, \ldots, w_{t_n}$ be a plane spanning path of P_n with one end in w_1. Since $w_1 w_{t_2}$ is a leaf of T, by Corollary 3, it must lie in $CH(P)$; without loss of generality we assume $w_{t_2} = w_2$. If $w_3 \neq w_{t_3} \neq w_n$, then w_3 and w_n are separated by the edge $w_2 w_{t_3}$. Since T is a plane spanning path of P_n, then w_3 is connected to w_{t_3} by a subpath of T contained in one of the half planes defined by the line l that contains the edge $w_2 w_{t_3}$, and w_n is connected to w_{t_3} by a subpath of T contained in the other half plane defined by l. This implies that at least one vertex of T must have degree 3 which is not possible since T is a path. Therefore either $w_{t_3} = w_3$ or $w_{t_3} = w_n$; this means that the first three vertices of T are consecutive vertices of P_n. By extending this argument, we can prove the following result.

Theorem 4. *Let T be a plane spanning path of P_n. For $k = 1, 2, \ldots, n$, the first k vertices of T are consecutive points of P_n.*

Let $X = \{x_1, x_2, \ldots, x_s\}$ be a set of points in convex position in the plane and let $\theta : X \to X$ be given by $\theta(x_k) = x_{s+1-k}$. For any geometric graph G with vertex set X, we denote by $\theta(G)$ the geometric graph, also with vertex set X, in which $\theta(x_i)$ and $\theta(x_j)$ are adjacent if and only if x_i and x_j are adjacent in G. In the case where X is a set of consecutive points of P_n, then $\theta(G)$ is the reflection of G with respect to the orthogonal bisector of the segment x_1x_s. The following lemmas will be used in the proof of Theorem 1; their proofs are omitted.

Lemma 5. *Let X be a set of points in convex position in the plane. If G is a plane geometric graph with vertex set X, then $\theta(G)$ is also a plane geometric graph with vertex set X.*

Lemma 6. *Let X be a set of points in convex position in the plane and let G and F be geometric graphs with vertex set X. If G and F are adjacent, then $\theta(F)$ and $\theta(G)$ are also adjacent.*

For any plane spanning path $T = w_{t_1}, w_{t_2}, \ldots, w_{t_n}$ of P_n and any integer $1 \leqslant k \leqslant n$, we denote by T_k the subpath $w_{t_1}, w_{t_2}, \ldots, w_{t_k}$ of T and by T^k the subpath $w_{t_k}, w_{t_{k+1}}, \ldots, w_{t_n}$ of T.

3. Proof of Theorem 1 for n odd

For $k = 0, 1, \ldots, n-1$ we show that there is a path J_k in G_n containing all plane spanning paths of P_n with middle point w_{k+1}, and that the paths $J_0, J_1, \ldots, J_{n-1}$ can be concatenated to give a Hamilton cycle of G_n.

For each $m \geqslant 1$, let us denote the points of P_{2m+1} as follows: For $i = 1, 2, \ldots, m+1$ let $u_i = w_i$ and $v_i = w_{2m-i+2}$.

Let A_m be the set of plane spanning paths of G_{2m+1} with middle point $u_{m+1} = w_{m+1} = v_{m+1}$, and denote by F_m the subgraph of G_{2m+1} induced by A_m. By Theorem 4, without loss of generality, we may assume $V(T_{m+1}) = \{u_1, u_2, \ldots, u_{m+1}\}$ and $V(T^{m+1}) = \{v_1, v_2, \ldots, v_{m+1}\}$ for every $T \in A_m$. We call T_{m+1} the *left subpath of* T and T^{m+1} the *right subpath* of T, and denote them by T_L and T_R, respectively (see Fig. 2). The following lemma is also presented without proof.

Lemma 7. *Two paths $T, S \in A_m$ are adjacent if and only if S_R and T_R are adjacent and $T_L = S_L$, or S_L and T_L are adjacent and $T_R = S_R$.*

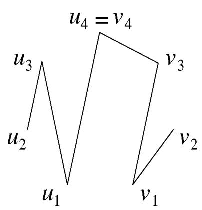

Fig. 2. $T_L = u_2, u_3, u_1, u_4$ and $T_R = v_4, v_3, v_1, v_2$.

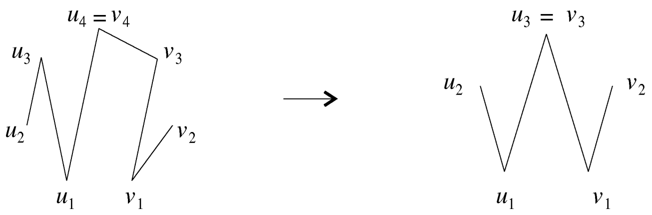

Fig. 3. $T = T_L * T_R \in A_3^{1,3}$ and $\alpha_{1,3}(T) = \theta^1(T_L - u_4) * \theta^0(T_R - v_4) \in A_2$.

Let L be a plane path with vertex set $\{u_1, u_2, \ldots, u_{m+1}\}$ and one end in $u_{m+1}$and R be a plane path with vertex set $\{v_1, v_2, \ldots, v_{m+1}\}$ and one end in v_{m+1}. We denote by $L * R$ the path in A_m with left subpath L and right subpath R.

For $i, j \in \{1, m\}$ let $A_m^{i,j}$ be the set of paths in A_m containing the edges $u_i u_{m+1}$ and $v_{m+1} v_j$, and denote by $F_m^{i,j}$ be the subgraph of F_m induced by $A_m^{i,j}$.

Lemma 8. *For $m \geqslant 2$ and $i, j \in \{1, m\}$, the graph $F_m^{i,j}$ is isomorphic to the graph F_{m-1}.*

Proof. For $i, j \in \{1, m\}$ let $\alpha_{i,j} : A_m^{i,j} \to A_{m-1}$ be the function given by $\alpha_{i,j}(T) = \theta^{\frac{m-i}{m-1}}(T_L - u_{m+1}) * \theta^{\frac{m-j}{m-1}}(T_R - v_{m+1})$, where $\theta^1 = \theta$ and θ^0 is the identity function (see Fig. 3).

Since $u_i u_{m+1}$ is an edge of T_L, then u_i is an end of $T_L - u_{m+1}$, and since $\theta^{\frac{m-i}{m-1}}(u_i) = u_m$, then u_m is an end of $\theta^{\frac{m-i}{m-1}}(T_L - u_{m+1})$. Since $v_{m+1} v_j$ is an edge of T, then v_j is an end of $T_R - v_{m+1}$, and since $\theta^{\frac{m-j}{m-1}}(v_j) = v_m$, then v_m is an end of $\theta^{\frac{m-j}{m-1}}(T_R - v_{m+1})$. Therefore $\alpha_{i,j}(T)$ is a plane spanning path of P_{2m-1} with middle point $u_m = w_m = v_m$.

If $\alpha_{i,j}(T) = \alpha_{i,j}(S)$, then $\theta^{\frac{m-i}{m-1}}(T_L - u_{m+1}) = \theta^{\frac{m-i}{m-1}}(S_L - u_{m+1})$ and $\theta^{\frac{m-j}{m-1}}(T_R - v_{m+1}) = \theta^{\frac{m-j}{m-1}}(S_R - v_{m+1})$. Since θ is a one to one function, then $T_L - u_{m+1} = S_L - u_{m+1}$ and $T_R - v_{m+1} = S_R - v_{m+1}$; this implies $T_L = S_L$ and $T_R = S_R$ and therefore $\alpha_{i,j}$ is also one to one.

For $T' = T'_L * T'_R \in A_{m-1}$ let $T_L = \theta^{\frac{m-i}{m-1}}(T'_L) + u_i u_{m+1}$ and $T_R = \theta^{\frac{m-j}{m-1}}(T'_R) + v_j v_{m+1}$. Since u_m is an end of T'_L and $\theta^{\frac{m-i}{m-1}}(u_m) = u_i$, then u_i is an end of $\theta^{\frac{m-i}{m-1}}(T'_L)$ and u_{m+1} is an end of T_L. Since v_m is an end of T'_R and $\theta^{\frac{m-j}{m-1}}(v_m) = v_j$, then v_j is an end of $\theta^{\frac{m-j}{m-1}}(T'_R)$ and v_{m+1} is an end of T_R. Therefore $T = T_L * T_R$ is a plane spanning path of P_{2m+1} with middle point $u_{m+1} = w_{m+1} = v_{m+1}$. Since $\theta^2 = \theta^0$ is the identity function, then $\alpha_{i,j}(T) = T'$ and therefore $\alpha_{i,j}$ is a function onto A_{m-1}.

By Lemmas 7 and 6, T and S are adjacent in $F_m^{i,j}$ if and only if $\alpha_{i,j}(T)$ and $\alpha_{i,j}(S)$ are adjacent in F_{m-1}. Therefore $F_m^{i,j}$ and F_{m-1} are isomorphic. $\square$

Proposition 9. *If $m \geqslant 2$, then F_m contains a Hamilton path J_m with ends $L_m * R_m$ and $L_m * R'_m$, where $L_m = u_1, u_2, \ldots, u_{m+1}$, $R_m = v_{m+1}, v_m, \ldots, v_1$ and $R'_m = v_{m+1}, v_1, v_2, \ldots, v_m$.*

Proof. The path J_2 is shown in Fig. 4. We proceed by induction assuming $m \geqslant 3$ and that the result holds for the graph F_{m-1}.

By induction F_{m-1} contains a Hamilton path J_{m-1} with ends in $L_{m-1} * R_{m-1}$ and $L_{m-1} * R'_{m-1}$. Since $F_m^{i,j}$ is isomorphic to F_{m-1}, then $F_m^{i,j}$ contains a Hamilton path $J_m^{i,j}$ with ends $\alpha_{i,j}^{-1}(L_{m-1} * R_{m-1})$ and $\alpha_{i,j}^{-1}(L_{m-1} * R'_{m-1})$. To end the proof we show how to concatenate the paths $J_m^{m,m}$, $J_m^{1,m}$, $J_m^{1,1}$ and $J_m^{m,1}$

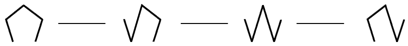

Fig. 4. The path J_2.

Fig. 5. $\lambda^0(L_4 * R_4')$ and $\lambda^5(L_4 * R_4)$.

to form a Hamilton path J_m of F_m with ends $\alpha_{m,m}^{-1}(L_{m-1} * R_{m-1}) = L_m * R_m$ and $\alpha_{m,1}^{-1}(L_{m-1} * R_{m-1}) = L_m * R_m'$.

Since $\alpha_{m,m}^{-1}(L_{m-1} * R_{m-1}') = (L_{m-1} + u_m u_{m+1}) * (R_{m-1}' + v_m v_{m+1})$ and $\alpha_{1,m}^{-1}(L_{m-1} * R_{m-1}') = (L_{m-1} + u_1 u_{m+1}) * (R_{m-1}' + v_m v_{m+1})$, then $\alpha_{1,m}^{-1}(L_{m-1} * R_{m-1}')$ can be obtained from $\alpha_{m,m}^{-1}(L_{m-1} * R_{m-1}')$ by deleting the edge $u_m u_{m+1}$ and adding the edge $u_1 u_{m+1}$. Analogously $\alpha_{1,m}^{-1}(L_{m-1} * R_{m-1})$ is adjacent to $\alpha_{1,1}^{-1}(L_{m-1} * R_{m-1})$ and also $\alpha_{1,1}^{-1}(L_{m-1} * R_{m-1}')$ is adjacent to $\alpha_{m,1}^{-1}(L_{m-1} * R_{m-1}')$. $\square$

We can now prove Theorem 1 for the case where $n = 2m + 1$.

Proof. Fig. 1 shows that G_3 is hamiltonian, therefore we may assume $m \geqslant 2$. For $k = 0, 1, \ldots, 2m$ let A^k be the set of plane spanning paths of P_{2m+1} with middle point w_{k+1} and F^k be the subgraph of G_{2m+1} induced by A^k.

Let $\lambda : P_{2m+1} \to P_{2m+1}$ be given by $\lambda(w_t) = w_{t+1}$. Since A^k is obtained from A^m by the rotation defined by λ^{k-m}, then F^k is isomorphic to F^m which is equal to F_m, as defined above. By Proposition 9, F^m contains a Hamilton path J_m with ends $L_m * R_m$ and $L_m * R_m'$. Therefore F^k contains a Hamilton path J_k with ends $\lambda^{k-m}(L_m * R_m)$ and $\lambda^{k-m}(L_m * R_m')$. To end the proof we show how to concatenate $J_0, J_1, \ldots, J_{2m}$ to obtain a Hamilton cycle of G_{2m+1}.

Let us call $\lambda^{k-m}(L_m * R_m)$ the left end of J_k and $\lambda^{k-m}(L_m * R_m')$ the right end of J_k. Since $(L_m * R_m' - w_{m+1} w_{2m+1}) + w_{2m+1} w_1 = \lambda^{m+1}(L_m * R_m)$, then $L_m * R_m'$ is adjacent to $\lambda^{m+1}(L_m * R_m)$ (see Fig. 5), thus the right end of J_m is adjacent to the left end of J_{2m+1}.We claim that for all t, the right end of J_t is adjacent to the left end of $J_{t+(m+1)}$, and therefore $J_m, J_{2m+1} = J_0, J_{3m+2} = J_{m+1}, J_{4m+3} = J_1, \ldots, J_{(2m+1)m+2m} = J_{2m}$ can be concatenated, in this order, to form a Hamilton cycle of G_{2m+1}. $\square$

4. Proof of Theorem 1 for n even

For each $m \geqslant 3$, let us denote the points in P_{2m} as follows: For $i = 1, 2, \ldots, m$ let $u_i = w_i$ and $v_i = w_{2m-i+1}$.

For $i, j \in \{1, m\}$, let $B_m^{i,j}$ be the set of plane spanning paths T of P_{2m} with middle edge $u_i v_j$, and such that $V(T^m) = \{u_1, u_2, \ldots, u_m\}$ and $V(T^{m+1}) = \{v_1, v_2, \ldots, v_m\}$. In this case, we call T^m the *left subpath* of T and T^{m+1} the *right subpath* of T, and denote them also by T_L and T_R, respectively. The following lemma is presented without proof.

Lemma 10. *Two paths $S, T \in B_m^{i,j}$ are adjacent if and only if S_R and T_R are adjacent and $T_L = S_L$ or S_L and T_L are adjacent and $T_R = S_R$.*

Let L be a plane path with vertex set $\{u_1, u_2, \ldots, u_m\}$ and one end in u_i $(i \in \{1, m\})$, and R be a plane path with vertex set $\{v_1, v_2, \ldots, v_m\}$ and one end in v_j $(j \in \{1, m\})$. We denote by $L * u_i v_j * R$ the path in $B_m^{i,j}$ with left subpath L and with right subpath R.

For $i, j \in \{1, m\}$, let $H_m^{i,j}$ denote the subgraph of G_{2m}, induced by $B_m^{i,j}$.

Lemma 11. *For $m \geqslant 3$ and $i, j \in \{1, m\}$ the graphs $H_m^{i,j}$ is isomorphic to F_{m-1}.*

Proof. For $i, j \in \{1, m\}$ let $\beta_{i,j} : B_m^{i,j} \to A_{m-1}$ be the function given by $\beta_{i,j}(T) = \theta^{\frac{m-i}{m-1}}(T_L) * \theta^{\frac{m-j}{m-1}}(T_R)$.

Since u_i is an end of T_L and $\theta^{\frac{m-i}{m-1}}(u_i) = u_m$, then u_m is an end of $\theta^{\frac{m-i}{m-1}}(T_L)$, and since v_j is an end of T_R and $\theta^{\frac{m-j}{m-1}}(v_j) = v_m$, then v_m is an end of $\theta^{\frac{m-j}{m-1}}(T_R)$. Therefore the middle point of $\beta_{i,j}(T)$ is $u_m = w_m = v_m$ which implies $\beta_{i,j}(T) \in A_{m-1}$.

If $\beta_{i,j}(S) = \beta_{i,j}(T)$, then $\theta^{\frac{m-i}{m-1}}(S_L) = \theta^{\frac{m-i}{m-1}}(T_L)$ and $\theta^{\frac{m-j}{m-1}}(S_R) = \theta^{\frac{m-j}{m-1}}(T_R)$. Since θ is a one to one function, then $S_L = T_L$ and $S_R = T_R$ and therefore $\beta_{i,j}$ is also one to one.

Let $T' = T'_L * T'_R \in A_{m-1}$. Since u_m is and end of T'_L and $\theta^{\frac{m-i}{m-1}}(u_m) = u_i$, then u_i is an end of $\theta^{\frac{m-i}{m-1}}(T'_L)$. Since v_m is an end of T'_R and $\theta^{\frac{m-j}{m-1}}(v_m) = v_j$, then v_j is an end of $\theta^{\frac{m-j}{m-1}}(T'_R)$. This implies $T = \theta^{\frac{m-i}{m-1}}(T'_L) * u_i v_j * \theta^{\frac{m-j}{m-1}}(T'_R) \in B_m^{i,j}$. Since $\theta^2 = \theta^0$ is the identity function, then $\beta_{i,j}(T) = T'$ and therefore $\beta_{i,j}$ is a function onto A_{m-1}.

By Lemmas 10 and 6, S and T are adjacent in $H_m^{i,j}$ if and only if $\beta_{i,j}(T)$ and $\beta_{i,j}(S)$ are adjacent in F_{m-1}. Therefore $H_m^{i,j}$ and F_{m-1} are isomorphic. □

Let $B_m = B_m^{m,m} \cup B_m^{1,m} \cup B_m^{m,1} \cup B_m^{1,1}$ and H_m be the subgraph of G_{2m}, induced by B_m (see Fig. 6). Let C be the cycle $u_1, u_2, \ldots, u_m, v_m, v_{m-1}, \ldots, v_1$ and S_m and S'_m be the paths $C - u_1 v_1$ and $C - u_m v_m$.

Proposition 12. *For $m \geqslant 3$ the graph H_m contains a Hamilton path I_m with ends S_m and S'_m.*

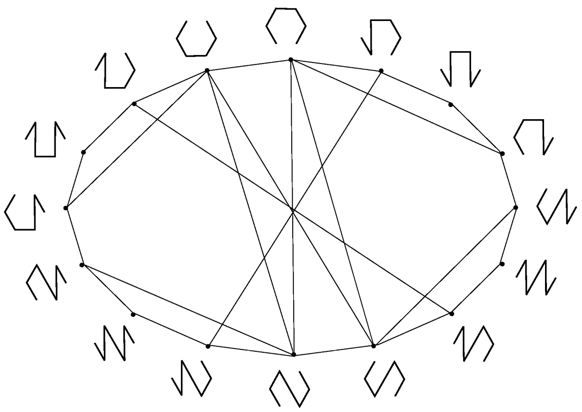

Fig. 6. The graph H_3.

Proof. By Lemma 11, each graph $H_m^{i,j}$ is isomorphic to F_{m-1} and by Proposition 9, F_{m-1} contains a Hamilton path with ends $L_{m-1} * R_{m-1}$ and $L_{m-1} * R'_{m-1}$. Therefore $H_m^{i,j}$ contains a Hamilton path $I_m^{i,j}$ with ends $\beta_{i,j}^{-1}(L_{m-1} * R_{m-1})$ and $\beta_{i,j}^{-1}(L_{m-1} * R'_{m-1})$. To end the proof we show how to concatenate $I_m^{m,m}$, $I_m^{1,m}$, $I_m^{m,1}$ and $I_m^{1,1}$ to form a Hamilton path I_m of H_m with ends $S_m = \beta_{m,m}^{-1}(L_{m-1} * R_{m-1})$ and $S'_m = \beta_{1,1}^{-1}(L_{m-1} * R_{m-1})$.

Since $\beta_{m,m}^{-1}(L_{m-1} * R'_{m-1}) = L_{m-1} * u_m v_m * R'_{m-1}$ and $\beta_{1,m}^{-1}(L_{m-1} * R'_{m-1}) = L_{m-1} * u_1 v_m * R'_{m-1}$, then $\beta_{1,m}^{-1}(L_{m-1} * R'_{m-1})$ can be obtained from $\beta_{m,m}^{-1}(L_{m-1} * R'_{m-1})$ by deleting the edge $u_m v_m$ and adding the edge $u_1 v_m$. Analogously $\beta_{1,m}^{-1}(L_{m-1} * R_{m-1})$ is adjacent to $\beta_{m,1}^{-1}(L_{m-1} * R_{m-1})$ and also $\beta_{m,1}^{-1}(L_{m-1} * R'_{m-1})$ is adjacent to $\beta_{1,1}^{-1}(L_{m-1} * R'_{m-1})$. $\square$

We can now prove Theorem 1 for the case $n = 2m$.

Proof. Fig. 1 shows that G_4 is hamiltonian, therefore we can assume $m \geqslant 3$. For $k = 1, 2, \ldots, 2m$ let B^k be the set of plane spanning paths T of P_{2m} with middle edge $w_i w_j$ (with $i \in \{k, k-m+1\}$ and $j \in \{k+1, k+m\}$) and such that $V(T^m) = [w_{k-m+1}, w_k]$ and $V(T^{m+1}) = [w_{k+1}, w_{k+m}]$. By symmetry $B^k = B^{k+m}$.

For $k = m, m+1, \ldots, 2m-1$ let H^k be the subgraph of G_{2m} induced by B^k and let $\mu : P_{2m} \to P_{2m}$ be given by $\mu(w_t) = w_{t+1}$. Since B^k is obtained from B^m by the rotation defined by μ^{k-m} then H^k is isomorphic to H^m which is equal to H_m, as defined above. By Proposition 12, H^m contains a Hamilton path I_m with ends S_m and S'_m; therefore H^k contains a Hamilton path I^k with ends $\mu^{k-m}(S_m)$ and $\mu^{k-m}(S'_m)$.

Since all paths $\mu^{k-m}(S_m)$ and all paths $\mu^{k-m}(S'_m)$ are obtained from the cycle $C = w_1, w_2, \ldots, w_{2m}$ by deleting an edge, then they are pairwise adjacent in G_{2m}. Therefore $I_m, I_{m+1}, \ldots, I_{2m-1}$ can be concatenated (in many ways) to obtain a cycle Q_{2m} of G_{2m} that contains every path in $B^m \cup B^{m+1} \cup \cdots \cup B^{2m-1}$. We claim Q_{2m} includes all plane spanning paths of G_{2m}.

Let T be a plane spanning path of G_{2m} with middle edge e. By Theorem 4 the first m vertices of T are consecutive points of P_{2m}; clearly the last m vertices of T are also consecutive points of P_{2m}. Therefore, there is an integer s with $1 \leqslant s \leqslant 2m$ such that $T - e$ consists of a subpath T_L with vertex set $[w_{s-m+1}, w_s]$ and a subpath T_R with vertex set $[w_{s+1}, w_{s+m}]$.

Since e is the middle edge of T, then e has an end in T_L and an end in T_R; let $1 \leqslant t \leqslant m$ be such that w_{s+t} is the end of e in T_R. Since $P' = V(T_L) \cup \{w_{s+t}\}$ is a set of points in convex position, $T_L + e$ is a plane spanning path of P' and e is a leaf of $T_L + e$, then e lies in $CH(P')$. Notice that w_{s-m+1} and w_s are the two points in P' which are adjacent to w_{s+t} in $CH(P')$, therefore one of them is the end of e in T_L. Analogously the end of e in T_R is one of the vertices w_{s+1} or w_{s+m}; therefore $e \in \{w_s w_{s+1}, w_s w_{s+m}, w_{s-m+1} w_{s+1}, w_{s-m+1} w_{s+m}\}$. Since T_L has vertex set $[w_{s-m+1}, w_s]$ and T_R has vertex set $[w_{s+1}, w_{s+m}]$, then $T = T_L * e * T_R \in B^s = B^{s+m}$. $\square$

5. Final remark

An anonymous referee noticed that for $n \geqslant 3$, the connectivity of G_n is equal to 2. This is because the zig-zag path $w_1, w_n, w_2, w_{n-1}, \ldots, w_{\lfloor n/2 \rfloor + 1}$ has degree 2 in G_n except when $n = 4$, but in this case Fig. 1 shows that G_4 also has connectivity equal to 2.

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